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Collatz Conjecture: Propositions derived from Parity Vector Analysis

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Abstract

This paper is based on the research of Riho Terras, Eric Roosendaal, and David C. Kay on the Collatz problem. It addresses the proposition that "every positive integer N > 1 has a finite stopping time" and uses Parity Vectors (PVs) to provide insight that will be useful in proving this proposition. First, we classify finite-length PVs into three categories based on Terras's convergence condition. Then, we generate and count PVs using our own method based on "the length of PV" and "the number of 1s in PV." Finally, we consider the results. We also propose a Bird's eye view of parity vectors as a tool to visually understand their behavior and convergence status. We analyze the set of parity vectors corresponding to the cosets that classify all integers, as well as the convergence status of characteristic parity vectors, based on actual measurement data. As a result, we list hypotheses that support the validity of the affirmative Collatz conjecture. The figures and tables in the main text can be viewed by clicking the corresponding hyperlinks and are summarized in Appendix 1. Additionally, demonstrations such as the parity vector Bird's eye view and the computerized data analysis can be accessed via the corresponding hyperlinks. The source text of the demonstration programs can be downloaded from the list in Appendix 2.

Keywords— Collatz Conjecture, Stopping Time, Total Stopping Time, Glide, Glide Record, Parity Vector, generator of Parity Vector, v Convergence time, Convergence Condition Formula , Diophantine equation, Bird's eye view of the parity vector

1 Introduction

In this chapter, we provide definitions of terms, notations, and lemmas, citing previous related research papers, as background knowledge for the discussion. In addition to these findings, we will explain our research results in detail from Chapter 2 onwards.

1.1 Collatz conjecture

The Collatz conjecture is the conjecture that "for any positive integer N, if N is odd, multiply by 3 and add 1, and if N is even, divide by 2, repeatedly, will result in 1." However, if N is odd, multiplying by 3 and adding 1 will always result in an even number, so further division by 2 does not change the meaning of the expectation.

Let $S_0 = N$, and for all i

$$S_{i+1} = \begin{cases} S_i/2 & if\ S_i\ \ is\ \ even \\ \\ (3 \cdot S_i + 1)/2 & if\ \ S_i\ \ is\ \ odd \end{cases} \tag{1-1}$$

This sequence of S_i, $S(N) = (S_0, S_1, S_2, \cdots, S_{i-1}, S_i, \cdots)$ is called the *Collatz sequence* of N.

1.2 Stopping Time and Total Stopping Time

For any positive integer N (=S_0), if there exists a smallest integer k in the Collatz sequence such that $S_k < S_0$, then k is called the *Stopping Time* (also called the *Glide*) of N. And if there exists q such that $S_q = 1$ is called the *Total Stopping Time* (also called *Delay*) of N. The Stopping Time of N is expressed as $\sigma(N)$, and the total stopping time of N is expressed as $\sigma_\infty(N)$.
In general, all positive integer N can be expressed in the form
$2^k m + r (0 \leq m, 0 \leq r < 2^k)$.
This means that the integer N is classified into 2^k modulo classes with 2^k as the modulo and r as the remainder. For convenience of explanation, the remainder class with 2^k as the modulo and r as the remainder is written as $N_r = \{2^k m + r\}$.

(The Stopping time of remainder class)
Here, if k=5, all integers can be classified into a set of 2^5(=32) cosets $N_r = \{2^5 m + r\}$ ($0 \leq r < 2^5$). Considering the stopping time for Nr of 32, we get Table 1.

Table 1: The Stopping Time of $N_r = \{2^5 m + r\}$ or (Appendix 1)

If r=11, $S_0 = 2^5 m + 11, S_1 = 3 \cdot 2^4 m + 17, S_2 = 3^2 \cdot 2^3 m + 26, S_3 = 3^2 \cdot 2^2 m + 13,$ $S_4 = 3^3 \cdot 2m + 20, S_5 = 3^3 m + 10$, and since $S_0 > S_5, \sigma(2^5 m + 11) = 5$.

That is, the stopping time for all integers belonging to Nr other than r=7,15,27,31 are all less than or equal to 5, and for r=7,15,27,31, stopping time, if it exists, is a finite integer greater than or equal to 6.

Lemma 1. There is no maximum value of stopping time.

Proof. If the maximum value exists, let it be k. Considering the integer $S_0 = 2^k - 1$,then $S_1 = 3 \cdot 2^{k-1} - 1, S_2 = 3^2 \cdot 2^{k-2} - 1, \cdots, S_k = 3^k - 1$, and $S_0 < S_i (1 \leq i \leq k)$, Stopping time must be greater than k. This contradicts the assumption that the maximum value is k. (Q.E.D.)

Lemma 2. An integer can be obtained with the given stopping time. But there is also a stopping time that does not exist.

Proof. To find an integer with the given stopping time , it can be obtained by referring to the Bird's eye view of the parity vector described later and using, for example, the parity vectors of the upper and lower limits of the un converged region. (See [Data Analysis 3 and 4] in Sections 3.2.2)

The method for finding all applicable integers is possible using the mathematical method of Theorem 1 in [4] or Theorem 2 in [5]. (Q.E.D.)

Lemma 3. If the proposition "*Every integer N>1 has a finite stopping time*" is true using this stopping time, then the Collatz conjecture "*all positive integers reach 1 (with total stopping time)*" is also true.

Proof. Explanation by the inductive method of the integer N.
First, the Collatz sequence of 2 becomes $\{2, 1\}$, which reaches 1. 3 is $\{3, 5, 8, 4, 2, 1\}$, 4 is $\{4, 2, 1\}$, 5 is $\{5, 8, 4, 2, 1\}, \cdots$.We can see that all numbers up to a certain measurable integer have a stopping time and reach 1 (have a total stopping time), i.e., the proposition is true.

Next, suppose that all integers less than or equal to N have a stopping time and reach 1. Consider N+1, and according to the premise of the proposition,N+1 has a certain stopping time k.
This means that the (k+1) th integer in the Collatz sequence of N+1 is an integer less than N+1(less than or equal to N). Then, from the induction assumption, all integers less than or equal to N reach 1 (have total stopping time), so N+1 reaches 1 (have total stopping time). Therefore, the proposition is true. (Q.E.D.)

Note that Riho Terras [1] proved that "*almost all integers N with N>1 has a finite Stopping Time*" (see Terras' theorem in [2]).

Demonstration 1. Click on Program 1 to check the Collatz sequence of integer N and stopping time.The source text of this program (PHP, Python) can be downloaded from Appendix 2.

1.3 Parity Vector and its Convergence

Parity vector v(N) for any positive integer N is defined as $v_i(N) = S_{i-1} \bmod 2$ $(1 \leq i)$. For k elements with $1 \leq i \leq k$, v(N) is described as v(N)=$(v_1, v_2, v_3, \cdots, v_k)$ or v(N)=$v_1 v_2 v_3 \cdots v_k$.
At this point, the number of elements k is called the *length of this parity vector* v(N) and the integer N(=S_0) that generated this parity vector v(N) is called a *generator* of v(N), S_{k-1} is called a *Pre-resultant* of v(N), and S_k is called a *resultant* of v(N). (References [6]).

(Example)
For N=17, the first six elements are $S_0 = 17, S_1 = 26, S_2 = 13, S_3 = 20, S_4 = 10, S_5 = 5$, so we find $v_1 = 1, v_2 = 0, v_3 = 1, v_4 = 0, v_5 = 0, v_6 = 1$. Therefore, we write v(17) = (1,0,1,0,0,1) or v(17) = 101001. 17 is the generator of this parity vector v(17), 5 is the pre-resultant, and the next $S_6 = 8$ is the resultant.

The parity vector completely describes the iterative operation of the Collatz operation of formula (1-1) on N. Below, we introduce some lemmas concerning the parity vector.([2][3][6]).

Lemma 4. If N is a positive integer of the form $2^k m + r (0 \leq r < 2^k)$, then the first k elements of the parity vector are dependent on only r.
(For the proof, see Lemma 1 in [2])

Lemma 5. Suppose $w_i (1 \leq i \leq k)$ is a parity vector $(w_1, w_2, \cdots, w_k)$ of length k. Then there exists some number N for which $v_i(N) = w_i (1 \leq i \leq k)$
(For the proof, see Lemma 2 in [2])

Lemma 6. Let $S_0 =$ N be a positive integer and v_i its parity vector.

Let $d(a,b) = \sum_i v_i(a \le i \le b)$. Especially, d(k) be a shorthand for $d(1,k)$.
That is, d(k) is the number of "1s" in the parity vector of length k.
Then $S_k \approx T_k = S_0 \cdot 3^{d(k)} 2^{-k}$ and $\lim(S_k - T_k)/S_k = 0$ for sufficiently large S_k.
(Or, in logarithmic form, $\log(S_k) \approx \log(T_k) = \log(S_0) + d(k) \cdot \log(3) - k \cdot \log(2)$)
(For the proof, see Lemma 4 in [2])

(Example)
Taking $N = S_0 = 2^{50} - 1 = 1125899906842623$ results in
$S_{50} = 3^{50} - 1 = 717897987691852588770248$ and
$T_{50} = S_0 \cdot 3^{50} \cdot 2^{-50} \approx 717897987691851951148749$
where the difference between the two numbers is already less than $10^{-15} \cdot S_{50}$.

Lemma 6 shows that we can estimate S_k from v(N), so we can use v(N) to consider the convergence of N. The convergence of N means whether or not N has a finite stopping time.

Now, let $v_i(1 \le i \le k)$ be a parity vector of length k, and for any $1 \le j \le k$, let c(j) be $c(j) = d(j) \cdot \log(3) - j \cdot \log(2)($ d(j) is the formula defined in Lemma 6).

In this case, if $c(j) < 0(i.e., T_j < S_0)$ for some j in $1 \le j \le k$, we call v *convergent*, and the smallest value j for which c(j) < 0 is called the *convergence time* of v, or more generally, the convergence time of any N with such a parity vector.
If there is no such value of j, we call it v *divergent* or v *un convergent*.

Here, $c(j) = d(j) \cdot \log(3) - j \cdot \log(2) < 0$, that is, the inequality

$$j > d(j) \cdot log(3)/log(2) \qquad (1-2)$$

are called *convergence condition formula*.
From the above, the convergence of parity vector $v_j(1 \le j \le k)$ with length k can be classified into the following three types.
① If $k > min\{\exists j \mid j > d(j) \cdot \log(3)/\log(2)\}$: *Already converged* (convergence time=j) (j <k)
② If $k = min\{\exists j \mid j > d(j) \cdot \log(3)/\log(2)\}$: *Just converged* (convergence time = k)
③ If $j < d(j) \cdot \log(3)/\log(2)$ for all j: *Un converged*

Hereafter, in order to shorten the text, words in the main text may be simplified as in the footnote.*
From formula (1-2), the relationship between d(j) and v convergence time when the length of PV is k is as shown an Table 2.
However, j is the minimum value that satisfies the convergence condition formula for values $j \le k$. The values of the number of convergence times in this table are consistent with the "values of the stopping time τ" in Table 3 in [1].

Table 2: Relationship between PV length k,d(j),and v convergence time or (Appendix 1)

Here, we understand that the stopping time (and glide) of an integer N and the convergence time of v(N) can be considered equivalent based on the considerations of Riho Terass [1] and Eric Roosendaal ([2],[3]). From now on, we will use the terms stopping time, glide, and convergence time as the same.

Lemma 7. Suppose n is the smallest generator of a parity vector V of length k with r number of 1s.

*(1) Parity Vector→ PV
(2) Already converged → A-conv., Just converged → J-conv., Un converged → U-conv.,
(3) Already converged PV → A-PV, Just converged PV → J-PV, Un converged PV → U-PV

Then find generator n' of parity vector $V \oplus x$ (x is 0 or 1) of length k+1. Where $V \oplus x$ is assumed to mean that x (0 or 1) is added after the parity vector V.
If the resultant of V matches x (that is $S_k(\text{mod } 2) = x$), then $n' = n$ is the solution.
But if S_k and x are mismatched (that is S_k (mod 2) $\neq x$), then $n' = n + 2^k$ is the solution.
Furthermore, if m is the resultant of V then $m' = m + 3^r$ is the pre-resultant of $V \oplus x$.
(See Section 2.2. For the proof, see Theorem A in [6])

1.4 Diophantine Equation

The existence of a generator of a given parity vector can be seen in Lemma 8 below.

Lemma 8. A parity vector of length k is uniquely generated by a positive integer less than or equal to 2^k . Then, multiple parity vectors of length k correspond one-to-one to their smallest generator.
(For the proof of Lemma 8, see Theorem B in [6])

(Find the generator of a parity vector)

Let a parity vector of length k with some integer n > 1 be v(n) = $(v_1, v_2, v_3, \cdots, v_k)$ and let d be the number of 1s in element v_i of v(n) and m be the resultant after $the operation of v(n)$ for n, then the following formula holds.
(See Theorem 4 in [4], Section 3 and 4 in [6])

$$m = (3^d/2^k)n + R \qquad (1-3)$$

However, R is considered to be the unique value of the parity vector v(n) and is calculated by the following formula (cf. chapter4 in [6].)

$$R = \sum_{i=1}^{k} v_i 2^{i-1} 3^{\delta(i)}/2^k \qquad \text{which} \quad \delta(i) = d - \sum_{j=1}^{i} v_j$$

By rearranging equation (1-3) and substituting $q = 2^k R$, the following equation is obtained.

$$2^k m - 3^d n = q \qquad (1-4)$$

Equation (1-4) uses m and n as variables, and since the coefficients 2^k and 3^d are relatively prime, it becomes a first-order *Diophantine equation*, and there are an infinite number of (m , n) solutions. Among these infinite solutions, n is a generator of v(n), and from Lemma 8, the value n_0 corresponding to $1 \leq n \leq 2^k$ is the minimum generator of the parity vector of length k and is uniquely determined. Correponding to n_0, the solution m_0 of resultant is also uniquely determined. The general solution for m and n is expressed as $n = n_0 + 2^k t, m = m_0 + 3^d t$.
t is a parametric variable and is an integer greater than or equal to 0.

Lemma 9. If the number of 1s in an un converged parity vector of length k is d, the generator n and resultant m of this PV can be found by solving the Diophantine equation (1-4).
Proof. As explained above.(Q.E.D.)

(Example)
(1) 1111000
$q = 2^0 \times 3^3 + 2^1 \times 3^2 + 2^2 \times 3^1 + 2^3 \times 3^0 = 65$
Equation $2^7 m - 3^4 n = 65$. Solution is n=15, m=10
General solution $n = 15 + 2^7 t, m = 10 + 3^4 t$
(2) 1110100
$q = 2^0 \times 3^3 + 2 \times 3^2 + 2^2 \times 3^1 + 2^4 \times 3^0 = 73$

Equation $2^7m - 3^4n = 73$.   Solution is n=7 , m=5
General solution $n = 7 + 2^7t, m = 5 + 3^4t$

(3) 110111001
$q = 2^0 \times 3^5 + 2^1 \times 3^4 + 2^3 \times 3^3 + 2^4 \times 3^2 + 2^5 \times 3^1 + 2^8 \times 3^0 = 1117$
Equation $2^9m - 3^6n = 1117$.   Solution is n= 219 , m=314
General solution $n = 219 + 2^9t, m = 314 + 3^6t$

According to Lemma 7 to 9, there are two ways to find the generator of an arbitrary PV of length k: using Lemma 7 to compute it sequentially from the first digit of the PV, or solving the Diophantine equation of Lemma 9.

1.5 Parity Vector Classification (J-conv., A-conv., U-conv.)

If the first number of PV is 0, the generator corresponding to that PV is even, so the stopping time is 1, and therefore the v convergence time is 1. The following PVs with a leading number of 1 are examined.
Based on the convergence condition formula (1-2) above, the convergence status of PVs of lengths 1 to 6 can be checked as shown in Table 3.
The log(3)/log(2) in the convergence condition formula was calculated as 1.58.
The “numbers in bold” in Table 3 are those that satisfy the convergence condition formula, and the “convergence status ” is based on the PV convergence classification explained in Section 1.3. Note that Just in “Convergence Status” means Just converged PV, Already means Already converged PV, and Blank means Un converged PV.

Table 3: Example of determining the convergence of a parity vector using a convergence condition formula or (Appendix 1)

Table 4: Examples of J-converged PV and Un converged PV by length or (Appendix 1)

2 Generating and Counting Parity Vectors

All parity vectors can be formally classified as follows: (1) *by length*, which ranges from 1 to infinity, and (2) *by the number of 1s* contained in the parity vector. Below, we will explain how to generate the J-PV and U-PV and how to calculate their number for each of (1) and (2).

2.1 Generating and Counting of Parity Vectors by Length

Based on the procedure of examining J-PV and U-PV by ”length” of parity vector in Table 3 and Table 4, you can learn how to generate PVs of length k+1 from PVs of length k and how to calculate their number.

2.1.1 Methods for Generating Just Converged PVs and Un Converged PVs by Length

It is clear that adding 0 or 1 to a J-PV or a A-PV of length k results in a A-PV of length k+1. Therefore, it can be seen that in order to generate a J-PV or a U-PV with length of k+1, it is sufficient to add 0 or 1 to a U-PV with length of k.

[**Algorithm 1**] Algorithm for generating Un converged PVs and Just converged PVs of length k+1 from Un converged PVs of length k.

(Explanation)
Let V be a U-PV of length k. We write $V \oplus x$ to add x(1 or 0) to V and d(k) the number of 1s in a PV of length k. The algorithm for finding U-PVs and J-PVs of length k+1 from U-PVs of length k and d(k) number of 1s is as follows. If there are multiple U-PVs of length k, it is performed for each of them.

Procedure 1: $V \oplus 1$ is a U-PV of length k+1.
Procedure 2: $V \oplus 0$ can be divided into the following two cases depending on whether the PV convergence condition formula (1-2) is satisfied.

> If $(k+1) > d(k+1) \cdot \log(3)/\log(2)$, it becomes J-PV of length k+1
> Otherwise, it is a U-PV of length k+1.

(Example)
$\log(3)/\log(2)$ is calculated as 1.58.

(1) Let V=11011 be one of the U-PVs of length 5.
Procedure 1: V⊕1=110111 is a U-PV with length 6
Procedure 2: V⊕0=110110 is not 6 > 4×1.58, so it is a U-PV with length 6,
Therefore, two U-PVs (110111, 110110) with length 6 are generated from a U-PV (11011) with length 5.

(2) Let V=110110 be one of the U-PVs of length 6.
Procedure 1: V⊕1=1101101 becomes U-PV with length 7
Procedure 2: V⊕0=1101100 is 7 > 4×1.58, so it is J-PV with length 7
Thus, one U-PV (1101101) and one J-PV (1101100) of length 7 are generated from U-PV (110110) of length 6.

The generators for the generated un converged PV and just converged PV can be found by sequentially calculating from the first digit of the PV as described in Section 1.3, or by solving the Diophantine equations as described in Section 1.4.

Using the above algorithm, the U-PV list in Table 5 and the J-PV list in Table 6 are created.

Table 5: List of Un-converged PVs by length (only a partial list) or (Appendix 1)

Table 6: List of J-converged PVs by length (only a partial list) or (Appendix 1)

2.1.2 Calculating the number of J-converged PVs and Un-converged PVs by "Length"

The calculation of the number of J-PVs and U-PVs per length of PV can be counted in the algorithm described in 2.1.1, but we formulate a recurrence formula for calculating the number of PVs and perform the calculation.

(1) Calculating the number of U-PVs by length

Since PVs with a leading 0 are J-PVs or A-PVs, PVs with a leading 1 are targeted. The condition for a PV of length k to be a U-PV is given by formula (1-2) with the direction of the inequality in the convergence condition formula changed, and the following function $\epsilon(k,d)$ is defined for k and d.

$$\epsilon(k,d) = \begin{cases} 1 & : if \quad k < d \cdot \log(3)/\log(2) \\ 0 & : if \quad other \end{cases}$$

Let W(k,d) be the number of U-PVs with length k and the number of 1s d. In this case, ① the number of U-PVs of length k+1 and d+1 when 1 is added to the U-PV of length k is W(k+1 ,d+1), and ② the number of U-PVs of length k+1 and d when 0 is added W(k+1 ,d) are expressed by the following formulae, respectively.

$$① \ W(k+1, d+1) = \epsilon(k+1, d+1) \cdot W(k, d)$$

$$② \ W(k+1, d) = \epsilon(k+1, d) \cdot W(k, d)$$

From the above two formulae, the number of U-PVs with length k+1, W(k+1,d), can be obtained by the following recurrence formula (2-1) using the number of U-PVs with length k, W(k,d).

$$W(k+1, d) = \epsilon(k+1, d)\{W(k, d) + W(k, d-1)\}, \qquad (2-1)$$

where the initial values are
W(1,0)=0, W(1,1)=1, and W(k,d)=0 (d>k).

From the above, the number of U-PVs of length k+1, W(k+1), is the sum of the results of (2-1).

$$W(k+1) = \sum_{d=a}^{b} W(k+1, d), \qquad (2-2)$$

where a=$\lceil (k+1) \cdot log(2)/log(3) \rceil$ ($\lceil y \rceil$ is the ceiling function) and b=k+1.

(2) Calculating the number of J-PV by length.

Let X(k,d) be the number of J-PVs with length k and the number of 1s d. The J-PVs of length k+1 can be found by adding one zero to the U-PVs of length k and checking whether the convergence condition formula (1-2) is satisfied. Therefore, let d be the number of 1s in the U-PV of length k, and define the function $\mu(k, d)$ for k and d as follows

$$\mu(k, d) = \begin{cases} 1 & : if \quad k > d \cdot \log(3)/\log(2) \\ 0 & : if \quad other \end{cases}$$

Then, the number of X(k+1, d) can be calculated using the number of U-PVs, W(k,d), and since J-PVs of length k+1 can be checked for convergence by adding 0 to U-PVs of length k, the following formula (2-3) is valid.

$$X(k+1, d) = \mu(k+1, d) \cdot W(k, d)(k \geq 1), \qquad (2-3)$$

where the initial values are X(1,0) = 1 and X(1,1) = 0.

From the above, the number of J-PVs of length k+1, X(k+1), is the sum of the results of (2-3).

$$X(k+1) = \sum_{d=a}^{b} X(k+1, d), \qquad (2-4)$$

where a=0 and b= $\lfloor (k+1) \cdot log(2)/log(3) \rfloor$ ($\lfloor y \rfloor$ is the floor function).

From the above, the following theorem holds.

Theorem 1. The number W(k+1) of un converged parity vectors of length k+1 generated from W(k) un converged parity vectors of length k can be found using formulae (2-1) and (2-2), and the number X(k+1) of Just converged parity vectors can be found using formulae (2-3) and (2-4).

Proof. As explained above.(Q.E.D.)

Using Theorem 1, the number of J-PVs, A-PVs, and U-PVs by length can be calculated as shown in Table 7.

Table 7: Summary table of J-converged, A-converged, and Un-converged PVs by PV length or (Appendix 1)

Note: (A) Total number of PVs(2^k), (B) Number of Just converged PVs, (C) Number of Already converged PVs, (D) Number of Un converged PVs, (E) Ratio of Un converged PVs D/A

Table 7 shows the values of "E: Un convergence ratio," which is the ratio of the total number of PVs of the same length to the number of U-PVs. As the length increases, the Un convergence ratio approaches zero as much as possible, which is consistent with the values of $|W_k|/|V_k|$ (divergence ratio) in Eric Roosendaal's paper [3] and with the correction value of "Table A. Values of the Distribution Function F(k)" in Riho Terras's paper [1].

The computer output of the number of pieces calculated by length can be seen by clicking on the link shown below.
[Data by length] [1 to 100] [1 to 1000] [1 to 10000] (Appendix 2)

2.2 Generating and Counting Just Converged PV and Un Converged PV by "Number of 1s"

Next, let us consider how to generate and calculate the number of J-PVs and U-PVs for each group with the same number of 1s in the parity vector (hereinafter referred to as "by number of 1s").

2.2.1 Method of generating Just converged PV and Un-converged PV by "Number of 1s"

Table 3 shows that the only U-PV with d=1 is V=1, the U-PV with d=2 is 11 and 110, the U-PV with d=3 is 111, 1110, and 1101, and the U-PV with d=4 is 1111, 11110, 111100, 11101, 111010, 11011, and 110110.
Similarly, Table 3 shows that there is one J-PV with d=1 for V=10, one J-PV with d=2 for 1100, and two J-PVs with d=3 for lengths of 5, 11100 and 11010. d=4 J-PVs are 1111000, 1110100, and 1101100. After d=5, U-PVs and J-PVs can still be generated based on the convergence condition formula (1-2), discriminating between J-conv. and U-conv. The generation algorithm can be thought of as follows.

[Algorithm 2] Algorithm to generate Un converged PV and J-converged PV with d+1 number of 1s from Un-converged PV with d number of 1s.
(Explanation)
Let V be a U-PV of length k and the number of 1s d. We write V$\oplus x$ to add x(1 or 0) to V. Also, A→B is used to mean that the number (or sequence of numbers) in A is replaced by B.
The algorithm 2 for finding d+1 U-PVs and J-PVs from a U-PV of length k and number of 1s d is as follows. If $V \oplus 0$ is a U-PV, then it is a U-PV with d number of 1s. Therefore, d+1 U-PVs can be generated by adding 0 to the PV of V⊕1.
When there are multiple U-PVs, this is done for each one.
Procedure 1: V⊕1 with d+1 1s added to V is a U-PV of length k+1
Replace V⊕1→V, d+1→d, and k+1→k.

Procedure 2: Also, V⊕0 of length k+1 is the PV with one zero added to V after Procedure 1 or Procedure 3.

Replace V⊕0→V and k+1→k.

Procedure 3: Check the convergence of the PVs generated in Procedure 2,

If $k > d \cdot log(3)/log(2)$ is satisfied, the PV is J-PV. The Procedure is terminated.

Otherwise, the PV is U-PV. Repeat Procedure 2.

(Example) Let V=11011 be the U-PV for k=5, d=4. The Procedure is described according to the above algorithm 2.

Procedure 1: Add 1 to V=11011. V⊕1=110111 is a U-PV of length k=6, d=5.

Let V=110111.

Procedure 2: Add 0 to V=110111 and set V⊕0=1101110 to V, k=7.

The number of 1s remains the same, 5.

Procedure 3: V=1101110 is a U-PV with k=7 and d=5 since 7< 5×1.58

Procedure 2: Add 0 to V=1101110 and set V⊕0=11011100 to V, k=8.

The number of 1s remains the same, 5.

Procedure 3: V=11011100 is a J-PV with k=8 and d=5 since 8> 5 × 1.58. Termination.

From the resulting U-PV 11011 with k=5, d=4, one can generate two U-PVs, 110111 with k=6 and 1101110 with k=7, and one J-PV, 11011100 with k=8. They are d=5 for the number of 1s.

The generators for the generated U-PV and J-PV can be found by sequentially calculating from the first digit of the PV described in Section 1.3 or by solving Diophantine equations as described in Section 1.4,

By executing algorithm 2 above , the U-PV list in Table 8 and the J-PV list in Table 9 below can be generated.

Table 8: Un converged PV list by "number of 1s" or (Appendix 1)

Table 9: J-converged PV list by "number of 1s" or (Appendix 1)

2.2.2 Calculating the Number of J-converged PVs and Un-converged PVs by "Number of 1s"

The calculation of the number of J-PVs and U-PVs by "number of 1s" can be performed in the algorithm 2 described in 2.2.1, but in this section, we will use a recurrence formula to calculate the number of PVs.

(1) Calculating the number of U-PVs by "number of 1s"
Since PVs with leading 0s are J-PVs or A-PVs, PVs with leading 1s are targeted.
Let W(d, u) be the number of U-PVs with d 1s and u 0s. To determine whether a PV is a U-PV, we define the following function $\epsilon(d,u)$ for d and u.

$$\epsilon(d,u) = \begin{cases} 1 & : if \quad (d+u) < d \cdot \log(3)/\log(2) \\ 0 & : if \quad other \end{cases}$$

Using this condition, we find the number of W(d+1, u) when the number of W(d, u) is known.
A U-PV with d+1 numbers of 1s can be obtained from a U-PV with d numbers of 1s and u numbers of 0s by adding 1 to the U-PV. Therefore, the number W(d+1, u) is expressed by the following formula ①.

$$① \ W(d+1,u) = \epsilon(d+1,u)\cdot W(d,u)$$

Furthermore, the number of PVs generated by adding multiple zeros to the U-PV of W(d+1,u) is expressed by the following formula ②.

$$② \ W(d+1,u+1) = \epsilon(d+1,u+1)\cdot W(d+1,u)$$

From the above formulae ① and ②, the number of U-PVs with d+1 number of 1s can be obtained from the following formula (2-5).

$$W(d+1,u) = \epsilon(d+1,u)\{W(d,u) + W(d+1,u-1)\}, \qquad (2-5)$$

where the initial values are
W(1,0)=1, W(1,1)=0, and W(d,u)=0 (u<0).

From the above, the total number of U-PVs with d+1 1s, W(d+1), is obtained by summing the results of formula (2-5).

$$W(d+1) = \sum_{u=a}^{b} W(d+1,u), \qquad (2-6)$$

where $a = 0$ and $b = \lfloor (d+1)\cdot\{log(3)/log(2) - 1\}\rfloor$ ($\lfloor y \rfloor$ is the floor function).

(2) Calculating the number of J-PVs by "number of 1s"
Next, the number of J-PVs with d+1 number of 1s is calculated.
The number of J-PVs can be obtained from the algorithm 2 for generating J-PVs and U-PVs in 2.2.1. A J-PV with d+1 1s can be obtained by adding 1 to a U-PV with d 1s, and then adding 0s until the convergence condition is satisfied.
Therefore, the number of J-PVs with d+1 1s, X(d+1), is equal to the total number of U-PVs with d 1s, W(d), and the following formula (2-7) is obtained.

$$X(d+1) = W(d) \qquad (2-7)$$

From the above, the following theorem holds.

Theorem 2. When W(d) un converged parity vectors with d numbers of 1s are generated, the number W(d+1) of un converged parity vectors with d+1 numbers of 1s can be found using formulae (2-5) and (2-6), and the number X(d+1) of just-converged parity vectors can be found using formula (2-7).

Proof. As explained above.(Q.E.D.)

From Theorem 2, if J-PV and U-PV are calculated for each number of 1s, they are shown in Table 10.
Note that among PVs of finite/infinite length with the same number of 1s, it is clear that all PVs except J-PV and U-PV are A-PV.

Table 10: Number of J-converged PV and Un-converged PV by "number of 1s" or (Appendix 1)

The computer output of the "number of 1s" count can be seen by clicking on the link shown below.
[Data by number of 1s] [1 to 100] [1 to 1000] [9000 to 10000] (Appendix 2)

3 Consideration of the Collatz Conjecture Using Parity Vector

Based on the results of parity vector and the characteristics of the parity vector described below, we describe the measurement data we obtained by using the computer, the development of tools for analyzing the data, and the results of data analysis. We hope that these results will help to solve the Collatz conjecture.

3.1 Graphical Representation of the Parity Vector and Setting the Parity Vector Characteristic Values

We will explain the Bird's eye view of the parity vector developed for the subsequent analysis and the indices introduced to represent the characteristic values of the PV.

3.1.1 Role of Bird's eye view of the Parity Vector

A graphical representation of the J-PV, U-PV, and A-PV trajectories would facilitate visual understanding and comprehension of the convergence status.
Therefore, a computer program with the following functions (1) and (2) was created and used as a tool for investigation and analysis work. This diagram showing the behavior of the parity vector is called the *Bird's eye view of the parity vector*.
(1) Display the trajectory of a given parity vector. Consider each partial PV whose length increases by one from the beginning of the PV and create the list of each characteristic value such as generator, stopping time (glide, same as v convergence frequency), ratio (stopping time / length of partial PV), etc.
(2) If a Collatz sequence of a given positive integer N has a stopping time, generate PVs until it reaching 1, and display the PV trajectory, stopping time and total stopping time.

[**Algorithm 3**] The algorithm for creating the skeleton of a Bird's eye view of a PV and displaying any parity vector of finite length is as follows.

(Explanation)
When the length of the PV is k and the number of 1s in the PV is d, the skeleton of the Bird's eye view can be created by referring to Table 2, which calculates the relationship between k, d, and the number of v convergences (stopping time) based on the convergence condition formula (1-2).

① Create a k-d coordinate cell table with the length k on the vertical axis and the number of 1s d on the horizontal axis, and enter "0" in the coordinate cell (k , d) corresponding to the pair of k and d where v converges for each PV length. We will call this cell the Just Converged Cell (JC Cell).

② The connection between the JC cells corresponding to each length is taken as the boundary, and the area to the right of this boundary is called the *un converged PV region*, and the area to the left of the boundary is called the *converged PV region*.

③ The method for plotting PV of any length is as follows.
First, if the first digit is "0", place a "0" in coordinate cell (1,0). If the first digit is "1", place a "1" in coordinate cell (1,1).

④ For the second digit and beyond, if it is a "0", place a "0" in the coordinate cell directly below, and if it is a "1", place a "1" in the coordinate cell diagonally down to the right. In general, if the coordinate where the kth digit is placed is (k , d), then if the k+1th digit is a "0", place a "0" in the coordinate cell directly below (k+1, d), and if it is a "1", place a "1" in the coordinate cell diagonally down to the right (k+1, d+1). Then repeat ④ until the last digit.

⑤ When all the digits of the PV are plotted, if the last number stops at the JC cell, it means that the PV is a converged PV (J-PV), and if it has not reached the JC cell, it is an un converged PV (U-PV). Furthermore, a PV that has passed the JC cell becomes an already converged PV (A-PV).

Figure 1 is an example of (1) and shows the trajectory of a PV= 1111011100010 of length 13. k in the left column of the table in Figure 1 indicates the length of the PV, and the number d in the heading indicates the cumulative number of 1s in the PV. Figure 2 shows a list of attribute value of partial PV by length from the beginning of the PV.

Figure 1: Example of a graphical representation of a Parity Vector sequence (Bird's eye view of the PV) or (Appendix 1)

Figure 2: Example of attribute value display by length of Parity Vector or (Appendix 1)

Figure 3 shows a J-PV of length 7, 1101100 (generator=59), a U-PV of length 11, 11011111010 (generator=27) and an A-converged PV of length 11, 11011010001 (generator=123) trajectory.

Figure 3: Trajectory of the J-converged, Un converged, and A-converged PV or (Appendix 1)

Demonstration 2. The graphical representation of the parity vector trajectory about (1) and partial PV attributes of the parity vector can be viewed by clicking on Program 2 .The source text of this program (PHP, Python) can be downloaded from Appendix 2.

Figure 4 is an example of (2).

Figure 4: Graphical representation of the parity vector sequence for a positive integer N or (Appendix 1)

Demonstration 3. A graphical representation of the trajectory of the parity vector of positive integer N in (2) can be seen by clicking on Program 3 .The source text of this program (PHP, Python) can be downloaded from Appendix 2.

3.1.2 PV characteristic value " PV Convergence Ratio" index

The following two indices are set as PV characteristic values as mentioned in the functional description (1) of the parity vector Bird's eye view generation program in 3.1.1.

(1) Convergence Ratio for Stopping Time(ST): Ratio of the stopping time (glide or v convergence times) of that PV to the length of the PV of an integer (generator).

ST convergence ratio = (Stopping time) /(Length of PV)

(2) Convergence ratio for total stopping time(TST): Ratio of the number of times the Collatz sequence reaches 1 (total stopping time or delay) to the length of PV of an integer (generator).

TST convergence ratio = (Total Stopping time)/(Length of PV)

These ratios compare "the length of PVs that converge to J-convergence" and "the length of PVs until convergence to 1" to "the length of PVs of a given length". Each of these ratios is a measure of how many times the length of the original PV converges. The larger the number, the longer the trajectory to convergence. These indicators are used in the subsequent data analysis. Below are examples of ST convergence ratios and TST convergence ratios.

(Example)

① The stopping time for 1101 (generator is 11) of length 4 is 5 and the total stopping time is 10, i.e., the length of J-convergence PV (11010) is 5. Therefore, the ST convergence ratio = 5/4 = 1.25 and the TST convergence ratio = 10/4 = 2.5.

② The stopping time for 11011 (generator is 27) of length 5 is 59 and the total stopping time is 70, i.e., the length of J-convergence PV is 59. Therefore, the ST convergence ratio = 59/5 = 11.8 and the TST convergence ratio = 70/5 = 14.

③ 111111111111111(generator is 32767) of length 15 has a stopping time of 51 and a total stopping time of 85, i.e., the length of J-convergence PV is 51. Therefore, the ST convergence ratio = 51/15 = 3.4 and the TST convergence ratio = 85/15 ≈ 5.67.

④ The stopping time for 110011101011 of length 12 (generator is 1491) is 4 and the total stopping time is 60, that is, PV length is 12, but convergence frequency is 4 A-PV, so the ST convergence ratio = 4/12 ≈ 0.333 is less than 1, and the TST convergence ratio = 60/12 = 5.

⑤ The Stopping time for 110111111010101000 (generator is 68891) of length 18 is 18 and the Total stopping time is 113, i.e., J-convergence PV of length 18. Therefore, ST convergence ratio = 18/18= 1, and TST convergence ratio = 113/18 ≈ 6.28.

As can be seen from the example above, a PV with an ST convergence ratio of 1 is a J-PV, a PV greater than 1 is a U-PV, and a PV less than 1 is an A-PV.

Note that the size of the ST convergence ratio is independent of the length of the PV.

3.2 Various characteristic Data on Parity Vector and the Results of their Analysis

3.2.1 Results of the calculation of the number of J-PV and U-PV by length

The number of J-PVs and U-PVs by length was calculated in Section 2.1.2, and Table 7 lists the ratio of Un converged PVs by length.
It can be seen that the ratio infinitely approaches zero as the PV length k is infinitely increased. Of course, since the lengths of the PVs under consideration become infinitely large, the ratio of Un converged PVs of a given length will never be zero.
However, we can provide some experimental data that support the fact proved by Riho Terras ([1]) that **"almost every integer N > 1 has a finite stopping time"** (see Terras' theorem in [2][3]). These data should also contribute to the resolution of the propositions **"Every integer N with N > 1 has a finite stopping time"**,which is equivalent to the Collatz conjecture.

These data are presented for analysis below.

[**Data Analysis 1**] PV Convergence Ratio of $N_r = \{2^5m + r\}$

All integers with N > 1 belong to some $N_r = \{2^5m + r\}(0 \le r < 2^5)$, which can be classified into 32 remainder classes by the remainder r. The stopping time of the numbers in each class is less than or equal to 5 for r other than r=7,15,27,31, as shown in Table 1 in Section 1.2. The calculations are omitted, but the ST convergence ratios of the PVs corresponding to each integer in those classes are all less than 1.

Next, the convergence ratios of the PVs corresponding to the integers in each class of r=7,15,27,31 are obtained, and the maximum ST convergence ratio and the maximum TST convergence ratio by length and the corresponding PV (generator) for each are summarized in Table 11 below. ($1 \leq k \leq 35$)

Table 11: Convergence Ratio for Numbers of $N_r = \{2^5 m + r\}$ (r=7, 15, 27 ,31) or (Appendix 1)

The bold numbers in the table indicate the maximum ST convergence ratio among the r=7, 15, 27 ,31 classes of each same length.
The plot of ST convergence ratios by r=7, 15, 27 ,31 classes is shown in Figure 5.

Figure 5: Graph of ST Convergence ratio for $N_r = \{2^5 m + r\}$ (r=7, 15, 27, 31) or (Appendix 1)

As can be seen in Figure 5, all the integers in the four classes of r=7, 15, 27, and 31 have stopping time, and the ST convergence ratio tends to increase slowly as the length increases from 5 or greater.

[**Data Analysis 2**] Glide Indicator Data
Next, the glide record and K-max-G(N) for glide (stopping time) measured by the computer, and the ST convergence ratio of these data are summarized at Table 12. The graph of ST convergence ratio of K-max-G(N) is shown in Figure 6.

Glide record is a measure defined by Eric Roosendaal ([2],[3]). Let G(N) denote the glide of a positive integer N. N is called a *Glide record* if G(M) < G(N) holds for all integers M such that M < N.
For example, integers 1 and 2 are obvious glide records. Others such as 3, 7, 27, and 703 of G(3)=4, G(7)=7, G(27)=59, and G(703)=81 are applicable. The glide record data in the Table3-2 were compiled by Eric Roosendaal from a compilation of measurements by several researchers and are available on the Internet.

K-max-G(N) is the maximum glide for each interval, measured for an integer N intervals $[2^k, 2^{k+1} - 1]$ $(k \geq 0)$ of the generator N with finer data than Glide Record. 40 powers or more of 2 are missing data, but data for some intervals have been shared from glide record.

The significance of the Glide data in Table 12 is that it implies that all integer values corresponding to PVs of length k between 1 and 61 converge, especially in each interval $[2^k, 2^{k+1} - 1] (0 \leq k \leq 61)$ where the existence of k-Max-G(N) (the maximum value of Glide) indicates that all integer values belonging to each interval converge (have finite Glide).

Table 12: ST Convergence Ratio of Glide Records and K-Max-G(N) or (Appendix 1)

Furthermore, from Figure 6, it can be inferred that the ST convergence ratio of K-max-G(N) for each section is approximately 18 or less and does not change rapidly and significantly with increasing PV length, but this cannot be theoretically guaranteed.

Figure 6: ST Convergence Ratio of K-Max-G(N) or (Appendix 1)

3.2.2 Results of J-PV and U-PV counts by "number of 1s"

In section 2.2, we generated k+1 J-PVs and U-PVs from k U-PVs with "number of 1s" and calculated the number of PVs.
The results confirmed the following facts.

(1) Some (but not all) U-PVs with "d number of 1s" converge to J-PVs in the same "d number of 1s" group.

(2) The number of J-PVs with "d+1 number of 1s" from formula (2-7) X(d+1)= W(d) is equal to the number of U-PVs with "d number of 1s". Therefore,

$$\sum_{d=1}^{n}(X(d+1) - W(d)) = 0 \qquad (2-8)$$

The above is explained using the example in Table 13.

Table 13: Relationship between J-converged PV and Un converged PV by "number of 1s" or (Appendix 1)

The PVs in the area enclosed by the bold line are PVs of (A) and (B) below.

(A) the three J-PVs of "4 numbers of 1" :
① 1101100 (59),② 1110100 (7), ③ 1111000 (15)

(B) the three types U-PVs of "4 numbers of 1" with different length:
ⓐ 11011(27),110110(59),ⓑ 11101(7), 111010(7), ⓒ 1111(15),11110 (15), 111100 (15)

All PVs other than those listed above with "4 numbers of 1" are A-PVs.
Incidentally, the U-PVs at ⓐ ,ⓑ , and ⓒ are PVs generated from the three U-PVs 1101, 1110, and 111 with "3 numbers of 1".

Here, U-PV 110110 (59) of ⓐ becomes J-PV 1101100 (59) of ① by adding 0. In other words, the 59 of the generator converges. However, the U-PV 11011 (27) of ⓐ is not a J-PV in the group of "4 numbers of 1". Similarly, the two U-PVs of ⓑ become the J-PVs of ②, and the three U-PVs of ⓒ become the J-PVs of ③.

In other words, in this example, six of the seven U-PVs with "4 numbers of 1" converge, but one does not.

In addition, as can be seen in the table, the number of J-PVs with "5 numbers of 1" is 7, which is the same as the number of U-PVs with "4 numbers of 1".

In conclusion, while it is not guaranteed that all U-PVs in a group with "d number of ones " will converge within the same group, there are as many J-PVs that converge within a PV group with "d + 1 number of ones" as there are such numbers.

Next, the PV data at the upper and lower limits of the un converged region of the above PV Bird's eye view are taken as a characteristic data analysis.

[Data Analysis 3] Upper Limit PV Data of the Un Converged Region

Since the PVs at the upper limit of the un converged region are all 1 PVs, generator can be expressed as $2^n - 1(n \geq 1)$ where $2^n - 1(n \geq 5)$ integers are the numbers belonging to $N_{31} = \{2^5m + 31\}$.

The ST convergence ratios for values of $1 \leq n \leq 10000$ and the figure plotting them are shown in Figure 7.
The ST convergence ratios for $2^n - 1$ PVs are within 3 to 5 for $n \geq 500$, resulting in a flat graph. All $2^n - 1$ integers are expected to have finite glide.

Figure 7: Graph of ST Convergence Ratio for $2^n - 1 (1 \leq n \leq 10000)$ or (Appendix 1)

Furthermore, as a method for finding integers with the specified stopping time (glide) described in Lemma 2, it is possible to use the upper limit PV of the un converged region. To find the J-PV of stopping time = k, first, let the number of consecutive 1s in the upper limit PV be d.
And using the convergence condition formula, calculate the maximum integer d that satisfies $k > d \cdot log(3)/log(2), i.e., \ d < k \cdot log(2)/log(3)$. Then, the desired J-PV will have d 1s followed by (k-d) 0s.
For example, if k=10, then d=6, so the candidate J-PV is 1111110000. The generator for this PV is 575, and the stopping time is 10. The generator can be calculated by sequential calculation using Lemma 7 or by solving Diophantine equations, as described in Section 1.4.

Demonstration 4. The above "upper limit PV method" can be used to "find an integer with a given stopping time (glide)" by clicking on Program 4.The source text of this program (PHP, Python) can be downloaded from Appendix 2.

When $M = 2^k - 1$, the following lemma holds for the two Collatz sequences when k is odd (k=2n-1) and when k is even (k=2n), where n is the same integer.

Lemma 10. If $M_1 = 2^{2n-1} - 1, M_2 = 2^{2n} - 1$, then there are elements of the same value in the two Collatz sequences of M_1 and M_2.
Proof. If $S_0(M) = 2^k - 1$, then repeating the Collatz operation k times gives $S_k(M) = 3^k - 1$.
① $S_0(M_1) = 2^{2n-1} - 1$ obtains by (2n-1) Collatz operations, $S_{2n-1}(M_1) = 3^{2n-1} - 1$.
Since $S_{2n-1}(M_1) = 3^{2n-1} - 1$ is an even number, by rearranging $(3^{2n-1} - 1)/2$ we get
$(3^{2n-1} - 1)/2 = (3 \times 3^{2(n-1)} - 1)/2 = \{2 \times 3^{2(n-1)} + (3^{2(n-1)} - 1)\}/2 =$
$3^{2(n-1)} + (3^{2(n-1)} - 1)/2$, where odd + even = odd.
Therefore, the following equation is true:
$S_{2n}(M_1) = (3^{2n-1} - 1)/2$ and
$S_{2n+1}(M_1) = \{3 \times ((3^{2n-1} - 1)/2) + 1\}/2 = ((3^{2n} - 3)/2 + 1)/2 = (3^{2n} - 1)/4$.
② On the other hand, $S_0(M_2) = 2^{2n} - 1$ becomes $S_{2n}(M_2) = 3^{2n} - 1$ after 2n Collatz operations.
$S_{2n}(M_2) = 3^{2n} - 1 = (3n + 1)(3n - 1)$ is an even number × even number, so
$S_{2n+1}(M_2) = (3^{2n} - 1)/2$, $S_{2n+2}(M_2) = (3^{2n} - 1)/4$.
From ① and ②, $S_{2n+1}(M_1) = (3^{2n} - 1)/4 = S_{2n+2}(M_2)$, and there are two Collatz sequences, M_1 and M_2, whose elements are the same.(Q.E.D.)

Lemma 10 states that if $M_1 = 2^{2n-1} - 1$ converges to 1 (total stopping time) when n is the same integer, then $M_2 = 2^{2n} - 1$ also converges to 1 (Total Stopping Time).This implies that the converse is also true.
If $M_1 = 2^{2n-1} - 1$ does not converge to 1 at infinity, then $M_2 = 2^{2n} - 1$ will not converge to 1 at infinity. The converse is also true.

[**Data Analysis 4**] Lower Limit PV Data of the Un Converged Region
We consider the algorithm to determine what type of string the PV at the lower limit of the un converged region will be.

[**Algorithm 4**] The PV of the lower limit of the un converged region can be found as follows.

(Explanation)
Assuming that the PV Bird's eye view of Figure 1 has been created in advance, the

coordinates of the cell with "length (k)" on the vertical axis and "number of 1s (d)" on the horizontal axis are denoted as (k,d).
In the following explanation, the Just Converged cell (cell with the number 0) in the PV Bird's eye view is abbreviated as the JC cell.

The PV of the lower limit of the un converged region starts by setting 1 to the cell coordinates (1,1) to the right of the JC cell (1,0).

After that, when the coordinates of the JC cell are (k, d), a 1 is placed in the cell to the right (k, d+1). For a row of length k+1 that does not have a JC cell, a 0 is placed in the cell (k+1, d+1) immediately below the cell (k, d+1) that has a 1 .

By continuing this operation, it is possible to create the PV of the lower limit of the un converged region.

(**Demonstration 5.**) The PV of the lower limit of the un convergence region can be created by clicking on Program 5.The source text of this program (PHP, Python) can be downloaded from Appendix 2.

If we use this operation to create a lower limit PV length of 1000, the string will look like this:

Lower Limit PV

1101101101011011010110110110101101101011011011010110110101101101011011010110110101101101101011011
0101101101101011011010110110110101101101011011011010110110101101101011011011010110110101101101011
0110101101101101011011010110110110101101101011011010110110110101101101011011011010110110101101101
0110110101101101101011011010110110101101101011011010110110101101101011011010110110101101101011101
1010110110101101101011011011010110110101101101101011011010110110101101101011011010110110101101101
1010110110110101101101011011011010110110101101101101011011010110110101101101101011011010110110101
1011010110110110101101101011011011010110110101101101011011011010110110101101101011011010110110110
1011011010110110110101101101011010110110101101101101011011010110110101101101011011010110110110110
1101011011010101101101011011011010110110101101101011011010110110101101101011011010110110110110110
1101011011011010110110101101101011011010110110101101101011010110110110101101101011011010110110110

Lemma 11. The generator of a PV whose lower limit of the un convergence region is k digits ($k \geq 5$) and whose length is 5 or more is an integer $2^5m + 27$.

Proof. The first 5 digit string of this PV is 11011, so its generator is 27. After that, 1 or 0 is added and the length increases by 1, so its generator can be expressed as $2^5m + 27$. Thus, we can see that the generator of each PV of length k (≥ 5) digits or more from the beginning of the lower limit PV belongs to $\{2^5m + 27\}(m \geq 0)$.(Q.E.D.)

In this data analysis, a PV of 10,000 digits in length is generated, the ST convergence ratios are calculated, and a plot of these ratios is shown in Figure 8.

Most of the ST convergence ratios for the lower PVs are within a factor of 2, and the plots for lengths greater than 10 are flat.

Figure 8: Graph of ST Convergence Ratio for Lower Limit PV (1 to 10000)
or (Appendix 1)

Furthermore, if you change the 1 at any position in the lower limit PV to 0, the PV up to that position (e.g., k) will converge and the stopping time will be k. Therefore, to find the Just PV with a stopping time of k, you simply change the 1 in the kth digit from the beginning of the lower limit PV to 0.

Demonstration 6. The above "lower limit PV method" can be used to "find an integer with a given stopping time (glide)" as stated in Lemma 2.
Click on Program 6 to check the specific Procedure.The source text of this program (PHP, Python) can be downloaded from Appendix 2.

4 Conclusion

The following is a summary of the results of the analysis of measured data that provides evidence for the positive conclusion of the proposition that "all PVs converge (have stopping time)" from the perspective of the Parity Vector Bird's eye view in 3.1 above. Below is our summary hypothesis, divided into two views: one from the perspective of PV length (k) on the vertical axis of the Bird's eye view, and the other from the perspective of "the number of 1s in PV (d)" on the horizontal axis of the Bird's eye view.

(1) Consideration from the perspective of PV length (k)
From small to large values of k (from top to bottom on the vertical axis), we can expect all PVs of the same length (corresponding generators) to have a finite Glide. The rationale for this is that, as shown in Table 7, the ratio of the number of U-PVs to the total number of PVs of the same length, "E: Un convergence Ratio" is the ratio of the number of U-PVs to the number of PVs of the same length, and as the length increases, the un convergence ratio approaches zero as far as it goes.

For example, for PVs of length 10000 (1 to 2^{10000} in generator), the un convergence ratio is $2.394397e-156$, which is an extremely small ratio. However, as the length $k \to \infty$, the rate of un convergence is never zero. The reason for this is that the PV of length k+1, which is the un converged PV of length k (generator=N) plus 1, is necessarily the un converged PV (generator is N or N+2^k). Therefore, the necessary investigation would be to know when, if ever, all un converged PVs of the same length converge. Although there is no mathematical or logical proof for this point, it can be inferred from the data analysis described above that "all un-converged PVs of length k (generator=N) converge at PVs of length $k + p(0 < p < \infty)$ (generator=N)".

① From the measured data of the maximum ST convergence ratio of PV by length for the four remainders of integer $N = 2^5 m + r$ with r=7, 15, 27, and 31 in [Data Analysis 1], we can confirm the tendency that "for all four remainders, the maximum ST convergence ratio increases gradually with finite size as the length increases."

② From the analysis of Glide Records and K-Max-G(N) index data in [Data Analysis 2], all un converged PVs of length k converge in PV groups of length $k + p(0 < p < \infty$, where k+p is the Glide record or k-Max-G(N) of the PV belonging to length k).

③ The upper limit of the un converged region $PV = 111111\cdots$ (generator is $2^k - 1, 0 < k$) in the PV bird's-eye view in [Data Analysis 3] can be inferred that its ST convergence ratio is between 3 and 5 when $k \geq 500$, as can be seen in Figure 7. Therefore, all $2^k - 1$ integers are expected to have finite Glide.

④ In the case of the lower limit of PVs in the un converged region in [Data Analysis 4], the ST convergence ratio for PVs of length 10 or more is less than 2. All generators corresponding to partial PVs in the lower limit of PVs are expected to have finite Glide.

(2) Consideration from the perspective of PVs with "d number of 1s"
A situation can be observed where all PVs with the same "number of 1s" converge as the value of d goes from small to large (from left to right on the horizontal axis). The rationale for this is that the number of U-PVs with "d number of 1s" is the same as the number of J-PVs with "d+1 number of 1s," as obtained from the calculation of the number of J-PVs and U-PVs in 3.2.2. Then, the J-PV with "d + 1 number of 1s" is the U-PV with "d + 1 number of 1s" plus some zeros. This implies that all U-PVs with a finite number of d converge. However, we would like to add that this fact would not hold if there exist infinitely divergent U-PVs.

Although we were not able to prove that the Collatz Conjecture holds positively, we believe that we were able to provide supporting data that the Collatz Conjecture would hold. We hope that our results will be useful information for researchers who aim to solve the Collatz Conjecture using the Parity Vector.

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APPENDIX 1

(Tables and Figures)

Table 1: The Stopping Time of $N_r = \{2^5 m + r\}$

r	even	1,5,9,13,17,21,25,29	3,19	11,23	7,15,27,31
σ (Nr)	1	2	4	5	6 or more (if it exists)

Table 2: Relationship between PV length k , d(j),and v convergence time

Length k	1	2	3	4	5	6	7	8	9	10
d(j)	0	1		2	3		4	5		6
Convergence times j	1	2		4	5		7	8		10
Length k	11	12	13	14	15	16	17	18	19	20
d(j)	6	7	8		9	10		11		12
Convergence times j		12	13		15	16		18		20
Length k	21	22	23	24	25	26	27	28	29	30
d(j)	13		14	15		16	17		18	
Convergence times j	21		23	24		26	27		29	
Length k	31	32	33	34	35	36	37	38	39	40
d(j)	19	20		21	22		23		24	25
Convergence times j	31	32		34	35		37		39	40
Length k	41	42	43	44	45	46	47	48	49	50
d(j)	25	26	27		28	29		30		31
Convergence times j		42	43		45	46		48		50

Table 3: Example of determining the convergence of a parity vector using a convergence condition formula

PV length ↓	Convergence time →	1	2		4	5	Convergence status of PV
	PV length →	1	2	3	4	5	(See note below)
1	0	**0**					Just
	1	1.58					
2	11	1.58	3.16				
	10	1.58	**1.58**				Just
3	111	1.58	3.16	4.74			
	110	1.58	3.16	3.16			
	101	1.58	**1.58**	3.16			Already
	100	1.58	**1.58**	**1.58**			Already
4	1111	1.58	3.16	4.74	6.32		
	1110	1.58	3.16	4.74	4.74		
	1101	1.58	3.16	3.16	4.74		
	1011	1.58	**1.58**	3.16	4.74		Already
	1100	1.58	3.16	3.16	**3.16**		Just
	1010	1.58	**1.58**	3.16	**3.16**		Already
	1001	1.58	**1.58**	**1.58**	**3.16**		Already
	1000	1.58	**1.58**	**1.58**	**3.16**		Already
5	11111	1.58	3.16	4.74	6.32	7.9	
	11110	1.58	3.16	4.74	6.32	6.32	
	11101	1.58	3.16	4.74	4.74	6.32	
	11011	1.58	3.16	3.16	4.74	6.32	
	10111	1.58	**1.58**	3.16	4.74	6.32	Already
	11100	1.58	3.16	4.74	4.74	**4.74**	Just
	11010	1.58	3.16	3.16	4.74	**4.74**	Just
	11001	1.58	3.16	3.16	**3.16**	**4.74**	Already
	10110	1.58	**1.58**	3.16	4.74	**4.74**	Already
	10101	1.58	**1.58**	3.16	**3.16**	**4.74**	Already
	10011	1.58	**1.58**	**1.58**	**3.16**	**4.74**	Already
	11000	1.58	3.16	3.16	**3.16**	**3.16**	Already
	10100	1.58	**1.58**	3.16	**3.16**	**3.16**	Already
	10010	1.58	**1.58**	**1.58**	**3.16**	**3.16**	Already
	10001	1.58	**1.58**	**1.58**	**1.58**	**3.16**	Already
	10000	1.58	**1.58**	**1.58**	**1.58**	**1.58**	Already

PV length ↓	Convergence time →	1	2		4	5		Convergence status of PV
	PV length →	1	2	3	4	5	6	(See note below)
6	111111	1.58	3.16	4.74	6.32	7.9	9.48	
	111110	1.58	3.16	4.74	6.32	7.9	7.9	
	111101	1.58	3.16	4.74	6.32	6.32	7.9	
	111100	1.58	3.16	4.74	6.32	6.32	6.32	
	111011	1.58	3.16	4.74	4.74	6.32	7.9	
	111010	1.58	3.16	4.74	4.74	6.32	6.32	
	111001	1.58	3.16	4.74	4.74	**4.74**	6.32	Already
	111000	1.58	3.16	4.74	4.74	**4.74**	**4.74**	Already
	110111	1.58	3.16	3.16	4.74	6.32	7.9	
	110110	1.58	3.16	3.16	4.74	6.32	6.32	
	110101	1.58	3.16	3.16	4.74	**4.74**	6.32	Already
	110100	1.58	3.16	3.16	4.74	**4.74**	**4.74**	Already
	110011	1.58	3.16	3.16	**3.16**	**4.74**	6.32	Already
	110010	1.58	3.16	3.16	**3.16**	**4.74**	**4.74**	Already
	110001	1.58	3.16	3.16	**3.16**	**3.16**	**4.74**	Already
	110000	1.58	3.16	3.16	**3.16**	**3.16**	**3.16**	Already
	101111	1.58	**1.58**	3.16	4.74	6.32	7.9	Already
	101110	1.58	**1.58**	3.16	4.74	6.32	6.32	Already
	101101	1.58	**1.58**	3.16	**4.74**	**4.74**	6.32	Already
	101100	1.58	**1.58**	3.16	**4.74**	**4.74**	**4.74**	Already
	101011	1.58	**1.58**	3.16	**3.16**	**4.74**	6.32	Already
	101010	1.58	**1.58**	3.16	**3.16**	**4.74**	**4.74**	Already
	101001	1.58	**1.58**	3.16	**3.16**	**3.16**	**4.74**	Already
	101000	1.58	**1.58**	3.16	**3.16**	**3.16**	**3.16**	Already
	100111	1.58	**1.58**	**1.58**	**3.16**	**4.74**	6.32	Already
	100110	1.58	**1.58**	**1.58**	**3.16**	**4.74**	**4.74**	Already
	100101	1.58	**1.58**	**1.58**	**3.16**	**3.16**	**4.74**	Already
	100100	1.58	**1.58**	**1.58**	**3.16**	**3.16**	**3.16**	Already
	100011	1.58	**1.58**	**1.58**	**3.16**	**4.74**	6.32	Already
	100010	1.58	**1.58**	**1.58**	**3.16**	**4.74**	**4.74**	Already
	100001	1.58	**1.58**	**1.58**	**3.16**	**3.16**	**4.74**	Already
	100000	1.58	**1.58**	**1.58**	**3.16**	**3.16**	**3.16**	Already

Note: Blanks of Convergence status indicate un converged PVs.

Table 4: Examples of J-converged PV and Un converged PV by length

PV length	1	2	3	4	5	6
Number of 1 in PV	0	1	1	2	3	3
v Convergence time	1	2		4	5	
Number of J-converged PVs	1	1	0	1	2	0
J-converged PV	0	10		1100	11100	
					11010	

PV length	1	2	3	4	5	6	
Number of Un converged PVs	1	1	2	3	4	8	
Un converged PV	1	11	111	1111	11111	111111	111110
			110	1110	11110	111101	111100
				1101	11101	111011	111010
					11011	110111	110110

Table 5: List of Un converged PVs by length (only a partial list)

PV length	1	2	3	4	5	6	7	8	9		10				11
v Convergence time	1	2		4	5		7	8			10				
Number of Un converged PVs	1	1	2	3	4	8	13	19	38		64				128
Un converged PV	1	11	111	1111	11111	111111	1111111	11111111	111111111	111111110	1111111111	1111111101	1111111110	1111111100	
			110	1110	11110	111101	1111011	11110111	111101111	111101110	1111011111	1111011101	1111011110	1111011100	abbreviation
				1101	11101	111011	1110111	11101111	111011111	111011110	1110111111	1110111101	1110111110	1110111100	
					11011	110111	1101111	11011111	110111111	110111110	1101111111	1101111101	1101111110	1101111100	
						111110	1111101	11111011	111110111	111110110	1111101111	1111101101	1111101110	1111101100	
						111100	1111001	11110011	111100111	111100110	1111001111	1111001101	1111001110	1111110100	
						111010	1110101	11101011	111010111	111010110	1110101111	1110101101	1110101110	1111111000	
						110110	1101101	11011011	110110111	110110110	1101101111	1101101101	1101101110		
							1111110	11111101	111111011	111111010	1111110111	1111110101	1111110110		
							1111010	11110101	111101011	111101010	1111010111	1111010101	1111010110		
							1110110	11101101	111011011	111011010	1110110111	1110110101	1110110110		
							1101110	11011101	110111011	110111010	1101110111	1101110101	1101110110		
							1111100	11111001	111110011	111110010	1111100111	1111100101	1111100110		
								11111110	111111101	111111100	1111111011	1111111001	1111111010		
								11110110	111101101	111101100	1111011011	1111011001	1111011010		
								11101110	111011101	111011100	1110111011	1110111001	1110111010		
								11011110	110111101	110111100	1101111011	1101111001	1101111010		
								11111010	111110101	111110100	1111101011	1111101001	1111101010		
								11111100	111111001	111111000	1111110011	1111110001	1111110010		

Table 6: List of J-converged PVs by length (only a partial list)

PV length	1	2	3	4	5	6	7	8	9	10	11
Number of 1 in PV	0	1	1	2	3	3	4	5	6	7	8
v Convergence time	1	2		4	5		7	8		10	
Number of J-converged PVs	1	1	0	1	2	0	3	7	0	12	0
J-converged PV	0	10		1100	11100		1111000	11111000		1111110000	
					11010		1110100	11110100		1111101000	
							1101100	11110010		1111100100	
								11101100		1111011000	
								11101010		1111010100	
								11011100		1111001100	
								11011010		1110111000	
										1110110100	
										1110101100	
										1101111000	
										1101110100	
										1101101100	

Table 7: Summary table of J-converged, A-converged, and Un converged PVs by PV length

PV length (k)	(A)*1	(B)	(C)	(D)	(E)
1	2^1	1	0	1	0.5
2	2^2	1	2	1	0.25
3	2^3	0	6	2	0.25
4	2^4	1	12	3	0.1875
5	2^5	2	26	4	0.125
6	2^6	0	56	8	0.125
7	2^7	3	112	13	0.1015625
8	2^8	7	230	19	0.07421875
9	2^9	0	474	38	0.07421875
10	2^10	12	948	64	0.0625
20	2^20	2662	1018596	27328	2.60620E-02
30	2^30	0	1060970550	12771274	1.189418E-2
40	2^40	8.202367E+08	1.092288E+12	6.402835E+9	5.823345E-03
50	2^50	2.483696E+11	1.121917E+15	3.734259E+12	3.316689E-03
60	2^60	0	1.150705E+18	2.216134E+15	1.922191E-03
70	2^70	1.331807E+17	1.179216E+21	1.241504E+18	1.051594E-03
80	2^80	6.202813E+19	1.208070E+24	8.032099E+20	6.643997E-04
90	2^90	0	1.237431E+27	5.085201E+23	4.107792E-04
100	2^100	3.205325E+28	1.267315E+30	3.025607E+26	2.386763E-04

PV length (k)	(A)	(B)	(C)	(D)	(E)
200	2^200	4.151050E+53	1.606933E+60	4.917911E+54	3.060424E-06
300	2^300	6.206542E+81	2.037036E+90	1.113588E+83	5.466708E-08
400	2^400	1.561289E+110	2.582250E+120	2.991976E+111	1.158670E-09
500	2^500	4.196042E+138	3.273391E+150	8.702128E+139	2.658445E-11
600	2^600	1.125789E+167	4.149516E+180	2.674561E+168	6.445478E-13
700	2^700	0	5.260138E+210	8.268337E+196	1.571887E-14
800	2^800	0	6.668014e+240	2.731440E+225	4.096331E-16
900	2^900	0	8.452712e+270	9.160496E+253	1.083734E-17
1000	2^1000	0	1.071508E+301	3.109438E+282	2.901926E-19
2000	2^2000	0	1.148130e+602	1.039984E+568	9.058070E-35
3000	2^3000	0	1.230231e+903	5.179976E+853	4.210569E-50
4000	2^4000	0	1.318204e+1204	3.103534E+1139	2.354366E-65
5000	2^5000	0	1.412467e+1505	2.031904E+1425	1.438550E-80
6000	2^6000	5.678168E+1709	1.513470E+1806	1.459296E+1711	9.642052E-96
7000	2^7000	4.795226E+1995	1.621696E+2107	1.074532E+1997	6.625977E-111
8000	2^8000	3.627867E+2281	1.737662E+2408	8.001078E+2282	4.604508E-126
9000	2^9000	2.909188E+2567	1.861919E+2709	6.127126E+2568	3.290758E-141
10000	2^10000	2.339473E+2853	1.995063E+3010	4.776973E+2854	2.394307E-156

(*1)The symbol ^ means exponentiation

Note:(A) Total number of PVs(2^k) (B) Number of Just converged PVs (C) Number of Already converged PVs
(D) Number of Un converged PVs (E) Ratio of Un converged PVs D/A

Table 8: Un converged PV list by "number of 1s"

Number of Ones	d = 1	d = 2	d = 3	d = 4	d = 5	d = 6		d = 7	d = 8	d = 9
Number of PVs	1	2	3	7	12	30		85	173	303
Un convergedPVs	1	11	111	1111	11111	111111	11110011			
		110	1110	11110	111110	1111110	111100110	abbreviation	abbreviation	abbreviation
			1101	111100	1111100	11111100	1110111			
				11101	111101	111111000	11101110			
				111010	1111010	1111101	111011100			
				11011	1111001	11111010	11101101			
				110110	111011	111110100	111011010			
					1110110	11111001	11101011			
					1110101	111110010	111010110			
					110111	1111011	1101111			
					1101110	11110110	11011110			
					1101101	111101100	110111100			
						11110101	11011101			
						111101010	110111010			
							11011011			
							110110110			

Table 9: J-converged PV list by "number of 1s"

NumberofOnes	d = 1	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7		d = 8	d = 9
Convergencetime	2	4	5	7	8	10	12		13	15
NumberofPVs	1	1	2	3	7	12	30		85	173
JustconvergedPVs	10	1100	11100	1111000	11111000	1111110000	110110110100	111100111000		
			11010	1110100	11110100	1111101000	110110111000	111101010100		
				1101100	11110010	1111100100	110111010100	111101011000	abbreviation	abbreviation
					11101100	1111011000	110111011000	111101100100		
					11101010	1111010100	110111100100	111101101000		
					11011100	1111001100	110111101000	111101110000		
					11011010	1101101100	110111110000	111110010100		
						1101110100	111010110100	111110011000		
						1101111000	111010111000	111110100100		
						1110101100	111011010100	111110101000		
						1110110100	111011011000	111110110000		
						1110111000	111011100100	111111000100		
							111011101000	111111001000		
							111011110000	111111010000		
							111100110100	111111100000		

Table 10: Number of J-converged PV and Un-converged PV by "number of 1s"

(A)	(B)	(C)	(D)
1	2	1	1
2	4	1	2
3	5	2	3
4	7	3	7
5	8	7	12
6	10	12	30
7	12	30	85
8	13	85	173
9	15	173	476
10	16	476	961
20	32	5936673	13472296
30	48	84141805077	248369601964
40	64	2029460152095008	6113392816333320
50	80	52028134169251235063	160509643506854706934
60	96	1325438036712274130536314	4327322846731848749589802
70	111	39402905856990930693224661645	90297035113874205499700937260
80	127	1176811162775483601693782177033888	2743802136233200494910736532266785
90	143	34088001847162371052513765703648936115	82583155244433866410001965296565270848
100	159	752276648034035903600966817196471179765521	2309006583627815515505755316801664143732646

(A)	(B)	(C)	(D)
200	317	6.568066E+86	1.529621E+87
300	476	6.501316E+131	2.014549E+132
400	634	1.041206E+177	2.435531E+177
500	793	1.356044E+222	4.209951E+222
600	951	2.538654E+267	5.947521E+267
700	1110	3.670484E+312	1.140475E+313
800	1268	7.396696E+357	1.734178E+358
900	1427	1.127316E+403	3.504613E+403
1000	1585	2.365970E+448	5.550533E+448
2000	3170	1.506626E+901	3.539938E+901
3000	4755	1.429200E+1354	3.393522E+1354
4000	6340	1.675502E+1807	3.980332E+1807
5000	7925	2.140600E+2260	5.096390E+2260
6000	9510	2.942851E+2713	7.008399E+2713
7000	11095	3.876301E+3166	9.522458E+3166
8000	12680	5.738736E+3619	1.410026E+3620
9000	14265	8.512281E+4072	2.099039E+4073
10000	15850	1.315414E+4526	3.244099E+4526

Note: (A) Number of 1 (B) Convergent time (C) Number of Just converged PVs (D) Number of Un converged PVs

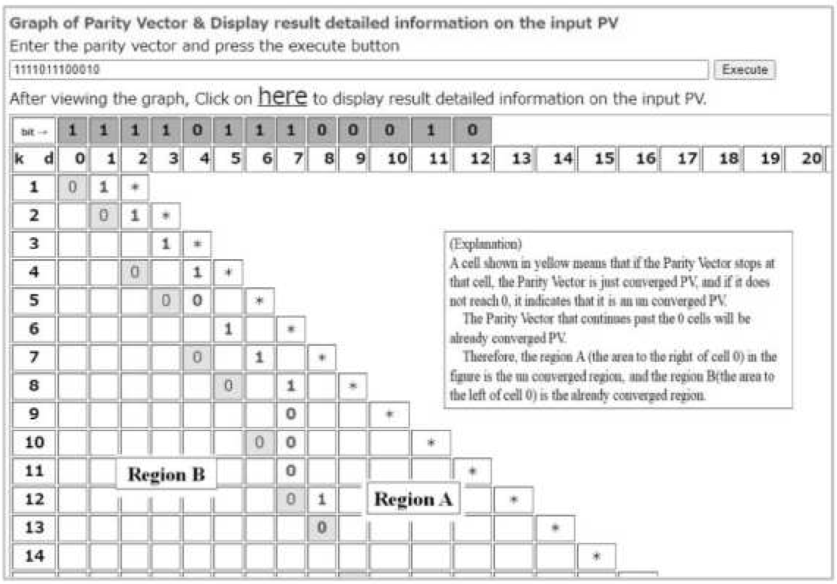
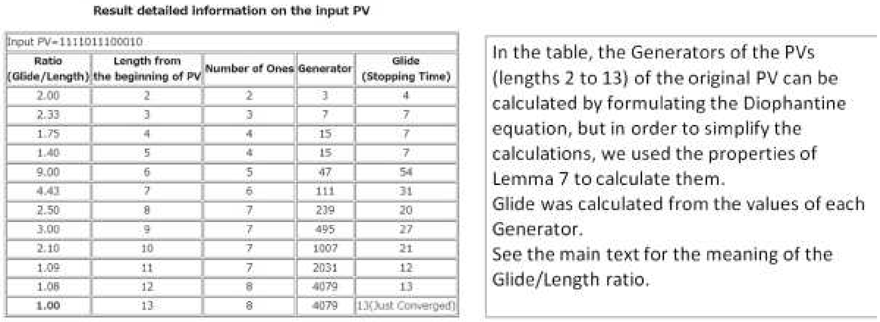


Figure 1: Example of a graphical representation of a Parity Vector sequence (Bird's eye view of the PV)



Result detailed information on the input PV

Input PV=1111011100010				
Ratio (Glide/Length)	Length from the beginning of PV	Number of Ones	Generator	Glide (Stopping Time)
2.00	2	2	3	4
2.33	3	3	7	7
1.75	4	4	15	7
1.40	5	4	15	7
9.00	6	5	47	54
4.43	7	6	111	31
2.50	8	7	239	20
3.00	9	7	495	27
2.10	10	7	1007	21
1.09	11	7	2031	12
1.08	12	8	4079	13
1.00	13	8	4079	13(Just Converged)

In the table, the Generators of the PVs (lengths 2 to 13) of the original PV can be calculated by formulating the Diophantine equation, but in order to simplify the calculations, we used the properties of Lemma 7 to calculate them.
Glide was calculated from the values of each Generator.
See the main text for the meaning of the Glide/Length ratio.

Figure 2: Example of attribute value display by length of Parity Vector

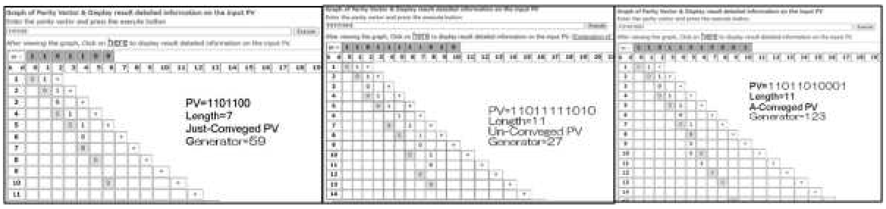


Figure 3: Trajectory of the J-converged, Un converged, and A-converged PV

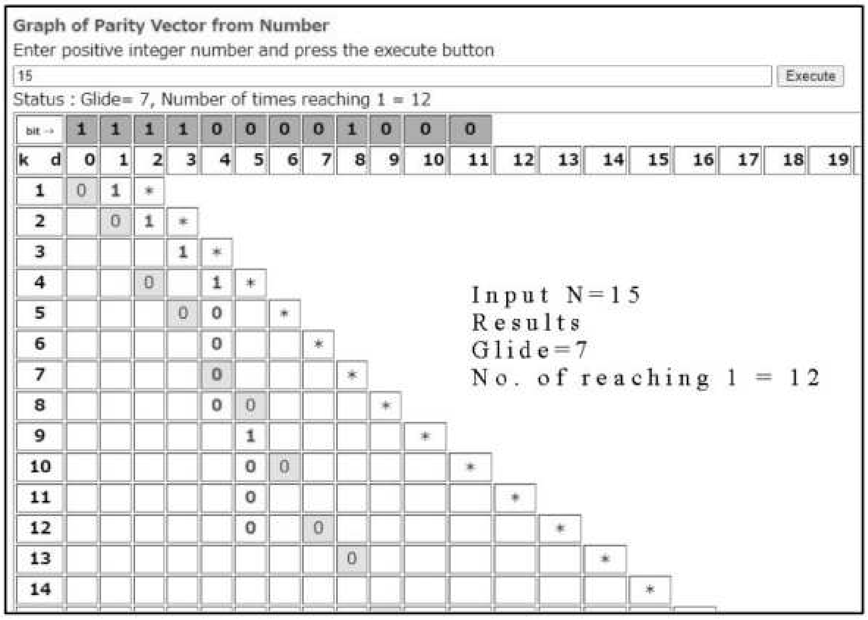


Figure 4: Graphical representation of the Parity Vector sequence for a positive integer N

Table 11: ST Convergence Ratio for Numbers of $N_r = \{2^5m+r\}$ (r=7, 15, 27, 31)

PV	Integer	N_7={2^5m+7}				N_{15}={2^5m+15}				N_{27}={2^5m+27}				N_{31}={2^5m+31}			
		ST Max		TST Max		ST Max		TST Max		ST Max		TST Max		ST Max		TST Max	
Length	No. of Total	Ratio	Generator	Ratio	Generator	Ratio	Generator	Ratio	Generator	Ratio	Generator	Ratio	Generator	Ratio	Generator	Ratio	Generator
3	1	**2.33**	**7**	3.67	7												
4	1					**1.75**	**15**	3.00	15								
5	2									**11.80**	**27**	14.09	27	11.26	31	17	31
6	4	1.31	39	3.81	39	**9.00**	**47**	11.00	47	1.17	59	3.67	59	9.00	63	11	63
7	8	**7.29**	**71**	8.26	71	3.43	111	8.43	111	3.43	91	8.43	91	2.13	127	10	95
8	16	3.61	167	10.13	231	2.30	239	7.13	207	**9.00**	**155**	6.68	155	3.88	223	8	223
9	32	2.33	327	10.11	327	3.00	495	9.11	463	2.87	283	9.44	411	**4.44**	**447**	8	415
10	64	[illegible]	871	11.38	871	2.16	751	9.48	879	2.70	795	8.70	763	**8.10**	**703**	11	703
11	128	3.91	1639	10.09	1895	7.09	1583	10.36	1743	6.95	1819	10.18	1307	**7.36**	**1407**	10	1695
12	256	4.00	3399	11.42	2919	4.17	2287	10.02	3695	5.90	3323	9.17	2843	**6.67**	**2111**	12	3711
13	512	5.62	7527	8.77	6375	**5.92**	**7279**	10.62	3823	5.53	7963	12.09	6171	5.28	4255	12	3903
14	1024	**7.50**	**10087**	10.21	14695	5.36	12399	11.86	12399	6.57	10131	10.07	10131	5.93	15039	12	15039
15	2048	4.97	18599	11.67	20471	4.97	17647	11.73	17647	4.87	24891	11.33	22843	**5.40**	**22559**	13	29823
16	4096	**8.44**	**35655**	12.79	30055	7.94	64975	11.38	52527	7.69	57310	12.08	57310	7.88	37503	13	59095
17	8192	6.47	110983	13.06	77031	6.65	126575	12.16	97087	6.12	116987	12.94	116947	**7.29**	**75007**	13	108159
18	16384	5.39	181319	15.44	238631	5.44	202751	13.58	218367	5.83	229587	15.11	142587	**6.89**	**206847**	13	158159
19	32768	8.58	382343	14.42	388181	7.21	438223	14.58	451263	7.86	422923	14.68	416011	**9.11**	**381727**	14	423679
20	65536	**9.15**	**1027431**	16.45	837799	8.75	667375	15.98	667375	8.80	626331	15.59	626331	8.50	1842431	15	1002463
21	131072	8.78	1349383	15.18	1349383	8.71	2054863	15.33	1590811	8.82	1943147	15.62	1256699	**10.67**	**1126015**	17	1723519
22	262144	8.23	2964775	16.64	2642807	8.68	4052639	15.27	4053639	7.93	4212379	15.68	2729059	**10.18**	**2252031**	16	3420767
23	524288	8.61	6078599	16.57	6355687	8.35	6684623	16.78	7332199	9.65	6631675	16.78	5646499	**10.70**	**8088063**	16	6649279
24	1048576	**11.96**	**13421671**	17.58	19791295	8.48	10147375	17.83	18587503	10.38	14578779	19.71	12489903	10.21	12131999	18	8400511
25	2097152	10.72	27208575	17.04	28020807	11.48	28842343	17.58	31468383	11.44	20132507	18.84	23841803	**11.92**	**26716671**	17	29638239
26	4194304	10.31	33873607	17.88	55147303	9.58	62512639	17.00	38794335	11.65	56624859	17.66	55624859	**14.46**	**63728127**	23	63728127
27	8388608	10.48	122159271	17.86	128207223	12.74	98883183	18.78	98883183	9.41	85437723	17.96	112517531	**13.63**	**95592191**	22	127456299
28	16777216	12.25	145224775	20.98	226456885	**14.11**	**217740015**	20.08	195877103	12.07	217987103	18.88	254877787	12.08	181200887	21	201104383
29	33554432	12.14	308610823	20.03	447517287	12.41	488034047	20.24	483656471	11.48	467953339	19.24	351149641	**12.52**	**363861375**	20	396779575
30	67108864	11.27	988697511	20.23	854943751	11.47	1013896495	19.41	884434575	11.68	801398811	18.23	1025330223	**11.90**	**656055295**	21	876817279
31	134217728	12.81	1806487687	20.12	1874833283	12.84	1200991791	19.27	1788312751	11.61	1110188933	19.73	1697609083	**13.97**	**1827397567**	20	1361234559
32	268435456	10.81	4015548263	18.72	3160004551	13.28	4131668827	20.41	3507086111	**13.97**	**2788008987**	20.3	2810742687	13.08	3136518111	18	3670825503
33	536870912	11.22	7889748023	21.38	7125491223	13.27	4704189167	21.88	4890328815	11.24	7932588283	20.68	4876895915	**13.33**	**6273020223**	20	5181210655
34	1073741824	**16.09**	**14500812391/ 12235060455**	21.89	14500812391	11.38	12638838879	21.18	12212032815	11.21	12027300539	20.71	11093039899	11.01	14884335615	22	13371184527
35	2147483648	15.37	20648854519	21.51	38085187887	15.51	24477120911	21.54	20058791791	**15.6**	**21751218587**	21.74	31094683223	15.17	28130034703	21.8	28742380655

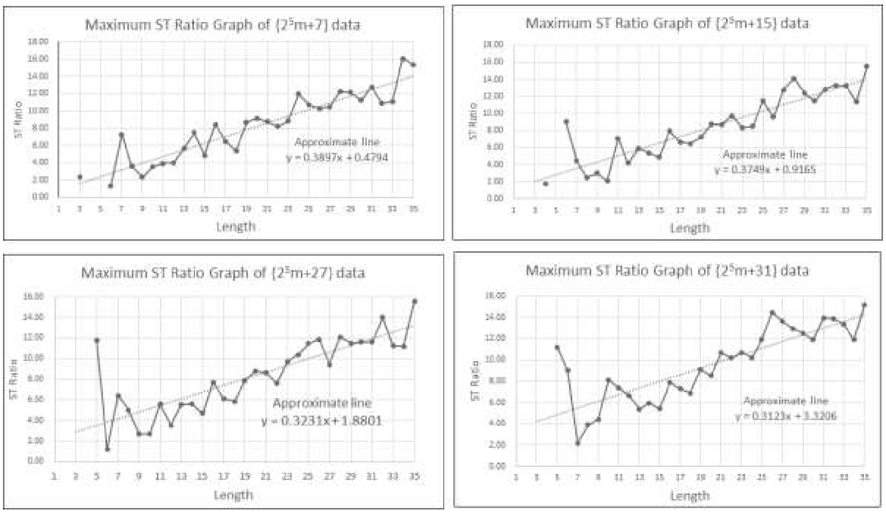


Figure 5: Graph of ST Convergence Ratio for $N_r = \{2^5m+r\}$ (r=7, 15, 27, 31)

Table 12: ST Convergence Ratio of Glide Records and K-Max-G(N)

Interval 2^k -	Generator Positive Integer N	Length of PV(LPV)	Glide Records Glide (*1)	K-Max-G(N) Glide (*2)	Convergence Ratio Glide/LPV
2^61	2602714556700227743	62	1005	1005	16.21
2^60	1236472189813512351	61	990	990	16.23
2^57	180352746940718527	58	966	966	16.66
2^56	118303688851791519	57	902	902	15.82
2^49	1008932249296231	50	886	886	17.72
	739448869367967	50	728		14.56
2^46	70665924117439	47	722	722	15.36
2^44	31835572457967	45	712	712	15.82
2^43	13179928405231	44	688	688	15.64
2^40	2081751768559	41	606	606	14.78
2^39	898696369947	40	550	550	13.75
2^38	377945493983	39		514	13.18
2^37	149311800091	38		520	13.68
2^36	83987887551	37		535	14.46
2^35	34980046495	36		463	12.86
2^34	21751218587	35		546	15.60
2^33	14500812391	34	547	547	16.09
2^32	6273020223	33		440	13.33
2^31	2788008987	32	447	447	13.97
2^30	1827397567	31	433	433	13.97
	1200991791	31	398		12.84
2^29	656055295	30		357	11.90
2^28	363861375	29		363	12.52
2^27	217740015	28	395	395	14.11
2^26	95592191	27		368	13.63
2^25	63728127	26	376	376	14.46
	56924955	26	308		11.84
2^24	26716671	25	298	298	11.92
	20638335	25	292		11.68

Interval 2^k -	Generator Positive Integer N	Length of PV(LPV)	Glide Records Glide (*1)	k-Max-G(N) Glide (*2)	Convergence Ratio Glide/LPV
2^23	13421671	24	287	287	11.96
2^22	8088063	23	246	246	10.70
2^21	2252031	22	224	224	10.18
2^20	1126015	21	224	224	10.67
2^19	1027431	20	183	183	9.15
2^18	381727	19	173	173	9.11
2^17	206847	18		124	6.89
2^16	75007	17		124	7.29
2^15	35655	16	135	135	8.44
2^14	22559	15		81	5.40
2^13	10087	14	105	105	7.50
2^12	7279	13		77	5.92
2^11	2111	12		80	6.67
2^10	1407	11		81	7.36
2^9	703	10	81	81	8.10
2^8	447	9		40	4.44
2^7	155	8		40	5.00
2^6	71	7		51	7.29
2^5	47	6		54	9.00
2^4	27	5	59	59	11.80
2^3	15	4		7	1.75
2^2	7	3	7	7	2.33
2^1	3	2	4	4	2.00

(*1) Editted by Eric Roosendaal in [2]:"3x+1 Glide Records"
http://www.ericr.nl/wondrous/glidrecs.html
(*2) K-Max-G(N) values without values in the missing interval [2^k,2^(k+1)-1] are missing data.

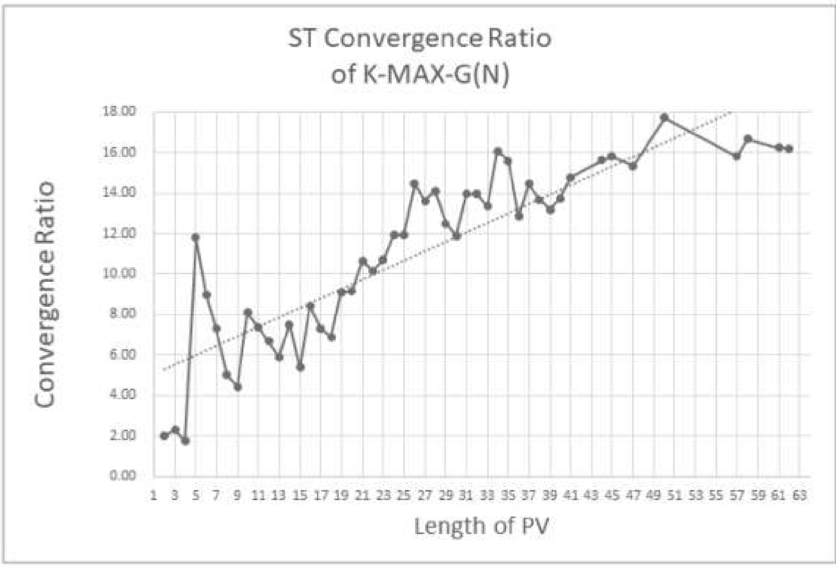


Figure 6: ST Convergence Ratio of K-Max-G(N)

Table 13: Relationship between J-converged PV and Un converged PV by "number of 1s"

Characters of Just converged Parity Vector (PV)				Un converged Parity Vector (PV)							
				Length	1	2	3	4	5	6	7
Parity Vector (PV)	Convergence time	Generator N	Number of 1s(Ones)	Number of PV → ↓	1	1	2	3	4	8	13
10	2	(1)	1	1	1(1)						
1100	4	(3)	2	2		11(3)	110(3)				
11010	5	(11)	3	1				1101(11)			
11100	5	(23)	3	2			111(7)	1110(7)			
1101100	7	(59)	4	2					11011(27)	110110(59)	
1110100	7	(7)	4	2					11101(7)	111010(7)	
1111000	7	(15)	4	3				1111(15)	11110(15)	111100(15)	
11011010	8	(123)	5	1							1101101(123)
11011100	8	(219)	5	2						110111(27)	1101110(91)
11101010	8	(199)	5	1							1110101(71)
11101100	8	(39)	5	2						111011(39)	1110110(39)
11110010	8	(79)	5	1							1111001(79)
11110100	8	(175)	5	2						111101(47)	1111010(47)
11111000	8	(95)	5	3					11111(31)	111110(31)	1111100(95)

(Note) The number in parentheses is generator of PV

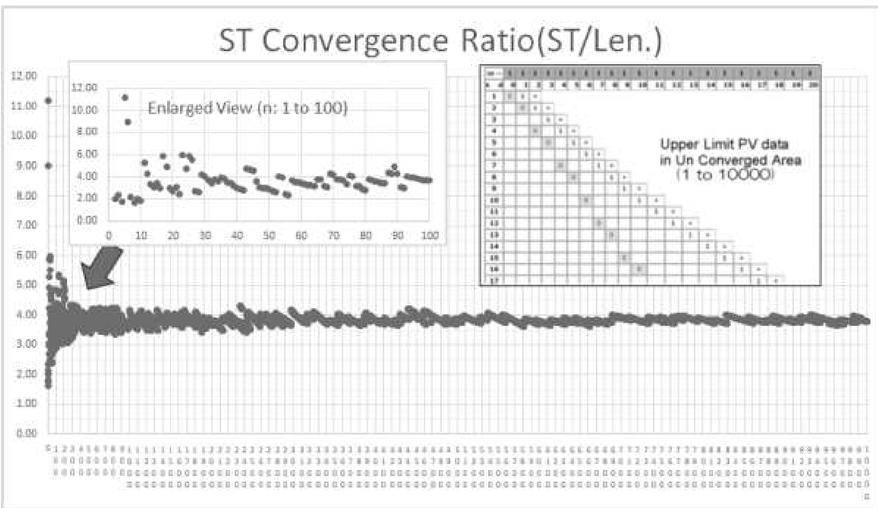


Figure 7: Graph of ST Convergence Ratio for Upper Limit PV Generator=$2^n - 1$ (n: 1 to 10000)

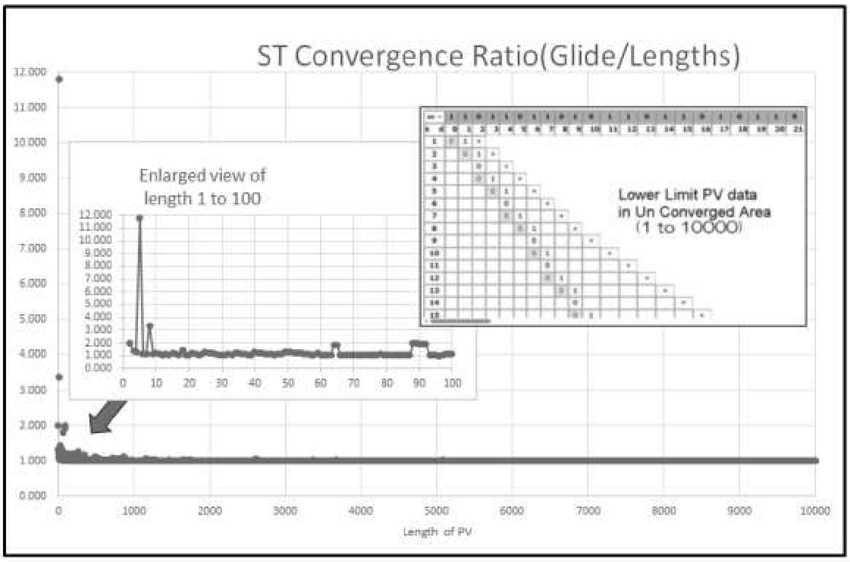


Figure 8: Graph of ST Convergence Ratio for Lower Limit PV (1 to 10000)

APPENDIX 2

(Computer programs and Data)

Following computer programs are the source texts of the demonstration programs linked to in the paper.
You can download the program you wish to use and modify it to suit your computer environment.

Table 14: List of Programs

	for PHP(*1) Language	for Python(*2) Language
Program 1	program1.html	program1py.html
	program1.php	program1.py
Program 2	program2.php	program2.py
Program 3	program3.php	program3.py
Program 4	program4.html	program4py.html
	Program4.php	program4.py
Program 5	program5.html	program5py.html
	Program5.php	program5.py
Program 6	program6.html	program6py.html
	Program6.php	program6.py
ViewOfBird.html	ViewOfBird.html is used in Program 2 and Program 3.	

(*1)PHP is an open-source server-side scripting language.
(*2)Python is a trademark of Python Software Foundation.

The following data are the textual data linked to in the paper.

Table 15: Data list of Numbers of PVs

	short size	middle size	long size
Data by length	1 to 100	1 to 1000	9000 to 10000
Data by number of 1s	1 to 100	1 to 1000	9000 to 10000

The demonstration programs in Table 1 were developed and are currently running in the following computer environment.

Table 16: Computer Environment (summary)

(1)	Computer	Windows Personal Computer
(2)	OS	Windows 10$^{(*1)}$
(3)	Server software	Xampp v3.3.0 $^{(*2)}$for Windows
(4)	Language	PHP 8 and Python 3
(5)	High precision integer	GMP function (PHP)
	calculations	mpmath (Python)

$^{(*1)}$ Windows is a trademark of Microsoft Corporation in United States.

$^{(*2)}$ XAMPP is a free and open-source cross-platform web server developed by Apache Friends.

[Acknowledgment]

The figures and tables in Appendix 1 and the programs and data in Appendix 2 are posted on GitHub.
GitHub is a software development platform operated by GitHub, Inc. in the United States.